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HÖLDER CONTINUITY OF SAMPLE PATHS OF SOME SELF-SIMILAR STABLE PROCESSES

by

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Technical Report No. 290

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HÖLDER CONTINUITY OF SAMPLE PATHS OF SOME SELF-SIMILAR STABLE PROCESSES

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Abstract.

The Hölder continuity of sample paths of the following classes of stochastic processes is examined: (1) Processes satisfying Kolmogorov's moment condition, (2) self-similar stable processes with stationary increments and (3) harmonizable fractional stable processes.

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1. Introduction and results

A stochastic process is said to be H-self-similar (H-ss) for $H>0$ if for any $c>0$, all finite-dimensional distributions of $\{X(ct)\}$ are the same as those of $\{c^H X(t)\}$, and to have stationary increments (si) if any finite-dimensional distribution of $\{X(t+b)-X(t)\}$ does not depend on b. It is also said to be α-stable if any finite-dimensional distribution of $\{X(t)\}$ is α-stable.

In this paper, we examine the Hölder continuity of H-ss si α-stable processes.

There are two main classes of H-self-similar α-stable processes with stationary increments: the linear fractional stable processes and the harmonizable fractional stable processes. In [T], Takashima showed the Hölder continuity of the linear fractional stable processes, and also pointed out that the exponent in the Hölder continuity cannot be bigger than $H-1/\alpha$. However, we can get a better Hölder continuity for the harmonizable fractional stable processes as follows. The harmonizable fractional stable process is a complex-valued process defined by

$$X(t) = \int_{-\infty}^{\infty} \frac{e^{it\lambda}-1}{i\lambda} |\lambda|^{1-H-1/\alpha} d\tilde{M}_\alpha(\lambda),$$

where $0<H<1$ and $\tilde{M}_\alpha$ is a complex rotationally invariant α-stable motion, (see [CM]). This is an H-ss si rotationally invariant α-stable process.

<u>Theorem 1</u>. For the harmonizable fractional stable process, there exists a version X^* such that

$$\lim_{\delta \downarrow 0} \sup_{|t-s|<\delta} \frac{|X^*(t)-X^*(s)|}{|t-s|^H |\log|t-s||^{1/\alpha+1/2+\epsilon}} = 0,$$

for any $\epsilon > 0$.

In [KM], we gave a partial result on the Hölder continuity of the harmonizable fractional stable process, where H is replced by any $\gamma < H$.

As mentioned above, Takashima [T] showed that the sample paths of the linear fractional stable process have the Hölder continuity of $|t-s|^{H-1/\alpha}$ with the help of some slowly varying function, if $1<\alpha<2$ and $1/\alpha < H < 1$. In the following theorem, we can see that this is also true for general H-ss si α-stable processes with $1<\alpha<2$ and $1/\alpha<H<1$.

Theorem 2. Let $X = \{X(t)\}_{t\geq 0}$ be H-ss si α-stable with $1<\alpha<2$ and $1/\alpha<H<1$. Then there exists a version X^* on $[0,1]$ such that

$$\lim_{\delta\downarrow 0} \sup_{\substack{|t-s|\in[0,1] \\ |t-s|<\delta}} \frac{|X^*(t)-X^*(s)|}{|t-s|^{H-1/\alpha} \quad |\log|t-s||^{1/\alpha+1+\epsilon}} = 0$$

for any $\epsilon > 0$.

α-stable processes have γ-th moments for any $\gamma < \alpha$. Then, by the property of H-ss si, we have

$$\begin{aligned} E[|X(t)-X(t)|^{\gamma}] &= E[|t-s|^{H\gamma}|X(1)|^{\gamma}] \\ &= C|t-s|^{H\gamma}. \end{aligned}$$

If $1<\alpha<2$ and $1/\alpha<H<1$, then we can find $1<\gamma<\alpha$ such that $H\gamma>1$. This means that H-ss si α-stable processes with $1<\alpha<2$ and $1/\alpha<H<1$ satisfy Kolmogorov's moment condition

$$E[|X(t)-X(s)|^{\gamma}] \leq K|t-s|^{H\gamma} \,, \tag{1.1}$$

where $\gamma > 1$, $K > 0$, $H\gamma > 1$. It is well-known that under condition (1.1), there exists a version whose sample paths are continuous. A natural question is whether we can get the Hölder continuity just under the moment condition (1.1) without self-similarity and stability. This is the case, as seen in the following theorem. This fact may be known but we cannot find appropriate references. We shall give its proof in this paper.

<u>Theorem 3</u>. Suppose $X = \{X(t)\}_{t\in[0,1]}$ satisfies

$$E[|X(t) - X(s)|^{\gamma}] \leq K|t-s|^{\beta} \,, \tag{1.2}$$

where $\gamma \geq 1$, $K > 0$, $\beta > 1$. Then there exists a version X^{*} such that

$$\lim_{\delta\downarrow 0} \sup_{\substack{|t-s|\in[0,1] \\ |t-s|<\delta}} \frac{|X^{*}(t)-X^{*}(s)|}{|t-s|^{(\beta-1)/\gamma}|\log|t-s||^{1+\epsilon}} = 0.$$

It is noted that Theorem 3 is not enough to get Theorem 2. If we apply Theorem 3 to H-ss si α-stable processes, we can only obtain

$$\lim_{\delta\downarrow 0} \sup_{|t-s|<\delta} \frac{|X^{*}(t) - X^{*}(s)|}{|t-s|^{\lambda}} = 0$$

for any $\lambda < H-1/\alpha$.

The proofs of Theorem 1, 2 and 3 are given in the subsequent sections.

2. Proof of Theorem 1

The basic idea to prove Theorem 1 is to use the LePage representation of complex-valued rotationally invariant stable processes. The LePage representation allows us to regard stable processes as conditionally Gaussian processes and we next use the known results for Gaussian processes.

We state results for the LePage representation and Gaussian processes as lemmas.

Let ψ be an arbitrary probability measure equivalent to Lebesgue measure on $\mathbb{R}$ and let φ be its Radon-Nikodym derivative, $\psi(d\lambda) = \varphi(d\lambda)d\lambda$. Let $\{\xi_i\}_{j\geq 1}$ be a sequence of iid random variables with distribution ψ, and let $\{g_j\}_{j\geq 1}$ be a sequence of iid rotationally invariant complex-valued random variables with $E[g_1] = 0$ and $E[|\mathrm{Re} g|^\alpha] = 1$. Let $\{\Gamma_j\}_{j\geq 1}$ be a sequence of Poisson arrival times with unit rate. Suppose that $\{\xi_i\}$, $\{g_j\}$, $\{\Gamma_j\}$ are mutually independent.

<u>Lemma 1</u>. Suppose $X = \{X(t)\}_{t\geq 0}$ is represented as

$$X(t) = \int_{-\infty}^{\infty} f(t,\lambda)d\tilde{M}_\alpha(\lambda).$$

Then $\{X(t)\}_{t\geq 0}$ has the same finite-dimensional distributions as $\{Y(t)\}_{t\geq 0}$ defined by

$$Y(t) = C \sum_{i=1}^{\infty} g_j \Gamma_j^{-1/\alpha} \varphi(\xi_j)^{-1/\alpha} f(t,\xi_j), \tag{2.1}$$

where the last series converges almost surefly for each t.

This result was shown in [MP]. However, there is a small gap in their proof, which is filled in [KM].

The next lemma due to [K1] was shown for real-valued processes, but it is easily seen to be valid also for the complex-valued case. More precisely, the lemma can be given from Theorem 1, Corollary 1 and the comment at the end of the proof of Theorem 1 of [K1].

<u>Lemma 2</u>. Let $\{Y(t)\}_{t\in[0,1]}$ be a centered Gaussian process satisfying

$$E[|Y(t)-Y(s)|^2] \leq \sigma^2(|t-s|),$$

where $\sigma(x)$ is a non-decreasing function defined on $(0,\infty)$ and $\sigma(x)|\log x|^{1/2}$ is also non-decreasing near the origin. Then

$$\lim_{\delta\downarrow 0} \sup_{|t-s|<\delta} \frac{|Y(t)-Y(s)|}{\sigma(|t-s|)\ |\log\ |t-s||^{1/2}} \leq \sqrt{2} \quad \text{a.s.}$$

<u>Proof of Theorem 1</u>. Recall that

$$X(t) = \int_{-\infty}^{\infty} f(t,\lambda)d\widetilde{M}_\alpha(\lambda),$$

where

$$f(t,\lambda) = \frac{e^{it\lambda}-1}{i\lambda} |\lambda|^{1-H-1/\alpha}.$$

Take

$$\varphi(\lambda) = \frac{a_\eta}{|\lambda||\log|\lambda||^{1+\eta}},$$

where $\eta > 0$ and a_η is the normalization for $\int\varphi(\lambda)d\lambda = 1$, and fix $\{\xi_i\}$ and $\{\Gamma_j\}$ in (2.1) to regard Y as a conditionally Gaussian process.

We denote the expectations with respect to $\{g_j\}$ and $\{\xi_j\}$ by E_g and E_ξ, respectively. In what follows, C denotes a positive constant which may differ from one inequality to another. We then have

$$\text{(2.2)}\quad E_g[|Y(t) - Y(s)|^2] = CE_g[|g_1|^2] \sum_{j=1}^{\infty} \Gamma_i^{-2/\alpha}\varphi(\xi_j)^{-2/\alpha}|f(t,\xi_j)-f(s,\xi_j)|^2$$

$$= Ca^2(|t-s|),$$

where

$$a^2(z) = \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}\varphi(\xi_j)^{-2/\alpha} \sup_{|t-s|<z} |f(t,\xi_j)-f(s,\xi_j)|^2$$

$$\leq C \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}\varphi(\xi_j)^{-2/\alpha}\{|z\xi_j|^2 \wedge 1\}|\xi_j|^{-2(H+1/\alpha)}.$$

Then we can prove

$$\text{(2.3)}\qquad \sum_{n=1}^{\infty} \frac{a^2(2^{-n})}{b^2(2^{-n})} < \infty \quad \text{a.s. } (\xi,\Gamma),$$

where

$$b(t) = t^H \left|\log|t|\right|^{(1+\eta)/\alpha}.$$

We are going to show (2.3). We have

$$E_\xi[a^2(z)] \le C \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \int_0^\infty \varphi(x)^{1-2/\alpha}\{|zx|^2 \wedge 1\}|x|^{-2(H+1/\alpha)}dx$$

$$= C \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}\{\int_0^{1/z} + \int_{1/z}^\infty\}$$

$$=: C \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \{I_1 + I_2\},$$

where

$$I_1 \le Cz^2\int_0^{1/z}(x|\log x|^{1+\eta})^{-(1-2/\alpha)}x^{-2(H+1/\alpha)+2}dx$$

$$= Cz^{2H} \ |\log z|^{-(1+\eta)(1-2/\alpha)}$$

and

$$I_2 \le C\int_{1/z}^\infty(x|\log x|^{1+\eta})^{-(1-2/\alpha)}x^{-2(H+1/\alpha)}dx$$

$$= Cz^{2H}|\log z|^{-(1+\eta)(1-2/\alpha)} .$$

Therefore

$$E_\xi[a^2(z)] \le (C \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha})z^{2H}|\log z|^{-(1+\eta)(1-2/\alpha)},$$

and thus

$$E_\xi[\sum_{j=1}^{\infty} \frac{a^2(2^{-n})}{b^2(2^{-n})}] \le (C \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}) \sum_{n=1}^{\infty} n^{-1-\eta} < \infty,$$

which is (2.3). From this,

$$\lim_{z\downarrow 0} \frac{a(z)}{b(z)} = 0 \quad \text{a.s. } (\xi,\Gamma),$$

which implies for small $z > 0$,

$$a(z) \le Cz^H\Big|\log |z|\Big|^{(1+\eta)/\alpha} \qquad \text{a.s. } (\xi,\Gamma) .$$

This combined with (2.2) gives us

$$E_g[|Y(t)-Y(s)|^2] \le C|t-s|^{2H} \Big|\log |t-s|\Big|^{2(1+\eta)/\alpha}.$$

If we regard this right-hand side as $\sigma^2(|t-s|)$ in Lemma 2, it satisfies the conditions in Lemma 2. Therefore by Lemma 2, almost surely with respect to (ξ,Γ)

$$\lim_{\delta\downarrow 0} \sup_{|t-s|<\delta} \frac{|Y(t)-Y(s)|}{|t-s|^H |\log|t-s||^{1/\alpha+1/2+\epsilon}} = 0$$

for any $\epsilon > 0$. The proof is thus completed. □

3. Proof of Theorem 2

We need a real-valued lemma.

<u>Lemma 3</u>. Let $\{f(t)\}_{t\in[0,1]}$ be a real-valued continuous function. Then we have

$$\sup_{|t-s|<2^{-n}} |f(t)-f(s)| \le 3 \sum_{r=n}^{\infty} \max_{1\le k\le 2^r} |f((k+1)2^{-r}) - f(k2^{-r})| .$$

<u>Proof</u>. Write the binary expansion of $t \in [0,1]$ as

$$t = \sum_{j=0}^{\infty} a_j(t)2^{-j}, \quad a_j(t) = 0 \text{ or } 1,$$

and put

$$t_r = \sum_{j=0}^{r} a_j(t)2^{-j} .$$

Then for t,s satisfying $|t-s| \le 2^{-n}$,

$$|f(t)-f(s)| \le \sum_{r=n}^{\infty} |f(t_{r+1})-f(t_r)| + |f(t_n)-f(s_n)| + \sum_{r=n}^{\infty} |f(s_{r+1})-f(s_r)|$$

$$\le 3 \sum_{r=n}^{\infty} \max_{1\le k\le 2^r} |f((k+1)2^{-r}) - f(k2^{-r})|.$$

This concludes the lemma. □

<u>Proof of Theorem 2</u>. As mentioned in Section 1, H-ss si α-stable processes with

$1 < \alpha < 2$ and $\frac{1}{\alpha} < H < 1$ satisfy moment condition (1.1). Hence there exists a version X^* with continuous sample paths. We write it X for simplicity of the notation. We restrict $X(t)$ on $\{t | t \in [0,1]\}$. Put

$$\Delta_n(X) = \max_{1 \leq k \leq 2^n} |X((k+1)2^{-n}) - X(k2^{-n})|.$$

By Lemma 3, we see

$$\sup_{|t-s|<2^{-n}} |X(t)-X(s)| \leq 3 \sum_{r=n}^{\infty} \Delta_r(X). \tag{3.1}$$

Let $\Phi(x)$ be a nonnegative, nondecreasing convex function defined on $[0,\infty)$ satisfying $\Phi(0) = 0$ and

$$\Phi(x) \sim \frac{x^\alpha}{(\log x)^{1+\eta}} \quad \text{as } x \longrightarrow \infty$$

for some η with $0 < \eta < \epsilon\alpha$. Denote the inverse function of $\Phi(x)$ by $\Phi^{-1}(x)$. $\Phi^{-1}(x)$ is a nonnegative, nondecreasing concave function on $[0,\infty)$ and satisfies

$$\Phi^{-1}(x) \sim \frac{1}{\alpha} x^{1/\alpha}(\log x)^{(1+\eta)/\alpha} \quad \text{as } x \longrightarrow \infty.$$

Since X is α-stable, we know

$$P\{|X(1)| > x\} \sim Cx^{-\alpha} \quad \text{as } x \longrightarrow \infty, \tag{3.2}$$

and therefore

$$E[\Phi(|X(1)|)] < \infty. \tag{3.3}$$

For simplicity, we put $\beta := H - \frac{1}{\alpha}$ (>0) below.

We now have

$$E\left[\sum_{n=1}^{\infty} \frac{\sup_{|t-s|\leq 2^{-n}} |X(t)-X(s)|}{2^{-n\beta} n^{1/\alpha+1+\epsilon}}\right] \leq 3 \sum_{n=1}^{\infty} \frac{1}{2^{-n\beta} n^{1/\alpha+1+\epsilon}} \sum_{r=n}^{\infty} 2^{-rH} E\left[\frac{\Delta_r(X)}{2^{-rH}}\right] \quad \text{(by (3.1))}$$

$$= 3 \sum_{n=1}^{\infty} \frac{1}{2^{-n\beta} n^{1/\alpha+1+\epsilon}} \sum_{r=n}^{\infty} 2^{-rH} E[\Phi^{-1} \circ \Phi(\frac{\Delta_r(X)}{2^{-rH}})]$$

$$\leq 3 \sum_{n=1}^{\infty} \frac{1}{2^{-n\beta} n^{1/\alpha+1+\epsilon}} \sum_{r=n}^{\infty} 2^{-rH} \Phi^{-1}(E[\Phi(\frac{\Delta_r(X)}{2^{-rH}})])$$

(by Jensen's inequality), where we have

$$E[\Phi(\frac{\Delta_r(X)}{2^{-rH}})] \leq \sum_{k=1}^{2^r} E[\Phi(\frac{|X((k+1)2^{-r})-X(k2^{-r})|}{2^{-rH}})]$$

$$= \sum_{k=1}^{2^r} E[\Phi(|X(1)|)] \qquad \text{(by H-ss si)}$$

$$= C2^r \qquad \text{(by (3.3))}.$$

Hence we have

$$E[\sum_{n=1}^{\infty} \frac{\sup_{|t-s|\leq 2^{-n}} |X(t)-X(s)|}{2^{-n\beta} n^{1/\alpha+1+\epsilon}}] \leq C \sum_{n=1}^{\infty} \frac{1}{2^{-n\beta} n^{1/\alpha+1+\epsilon}} \sum_{r=n}^{\infty} 2^{-rH} \Phi^{-1}(C2^r)$$

$$\leq C \sum_{n=1}^{\infty} \frac{1}{2^{-n\beta} n^{1/\alpha+1+\epsilon}} \sum_{r=n}^{\infty} 2^{-rH} 2^{r/2} (\log 2^r)^{(1+\tau_{\prime})/\alpha} \quad \text{(by (3.2))}$$

$$\leq C \sum_{n=1}^{\infty} \frac{1}{2^{-n\beta} n^{1/\alpha+1+\epsilon}} 2^{-n(H-1/\alpha)} n^{(1+\eta)/\alpha} < \infty,$$

implying

$$\sum_{n=1}^{\infty} \frac{\sup_{|t-s|\leq 2^{-n}} |X(t)-X(s)|}{2^{-n\beta} n^{1/\alpha+1+\epsilon}} < \infty \qquad \text{a.s.}$$

Therefore, there exists an N such that for any $n \geq N$,

$$\text{(3.4)} \qquad \sup_{|t-s|\leq 2^{-n}} |X(t)-X(s)| < 2^{-n\beta} n^{1/\alpha+1+\epsilon} \qquad \text{a.s.}$$

For any t,s satisfying $|t-s| < 2^{-N}$, take $n \geq N$ such that $2^{-n} \leq |t-s| < 2^{-n+1}$.

Then we have by (3.4)

$$|X(t)-X(s)| < |t-s|^{\beta}\Big|\log|t-s|\Big|^{1/\alpha+1+\epsilon} \qquad \text{a.s.}$$

and hence

$$\sup_{|t-s|\leq 2^{-N}} \frac{|X(t)-X(s)|}{|t-s|^{\beta}|\log|t-s||^{1/\alpha+1+\epsilon}} < 1 \qquad \text{a.s.}$$

Since $\epsilon > 0$ can be arbitrarily taken, we conclude Theorem 2. □

4. Proof of Theorem 3

We again start with the preliminary lemmas.

Lemma 4. Let $\{f(t)\}_{t\in[0,1]}$ be a real-valued continuous function and let

$$\Delta_n(f) = \max_{0\leq k\leq 2^n} \sup_{0\leq h\leq 2^{-n}} |f(k2^{-n}+h) - f(k2^{-n})| .$$

Let $\varphi(t)$ be a continuous function on $[0,1]$ such that $\varphi(0) = 0$ and $\varphi(t)$ is monotone increasing in the neighborhood of the origin. If

$$\sum_{n=1}^{\infty} \frac{\Delta_n(f)}{\varphi(2^{-n-1})} < \infty, \tag{4.1}$$

then

$$\lim_{\delta\downarrow 0} \sup_{|t-s|<\delta} \frac{|f(t)-f(x)|}{\varphi(|t-s|)} = 0.$$

Proof. For any $t,s \in [0,1]$, choose $n \in \mathbb{N}$ such that

$$2^{-n-1} \leq |t-s| < 2^{-n} ,$$

and represent t and s by

$$t = k_t 2^{-n} + h_t \quad \text{and} \quad s = k_s 2^{-n} + h_s,$$

respectively, where $k_t, k_s \in \mathbb{Z}$, $0 \leq h_t, h_s < 2^{-n}$. Noting $|k_t-k_s| = 0$ or 1, we have

$$(4.2) \quad |f(t)-f(s)| \leq |f(t)-f(k_t2^{-n})| + |f(k_t2^{-n})-f(k_s2^{-n})| + |f(k_s2^{-n})-f(s)|$$

$$\leq 3\Delta_n(f).$$

For $\delta>0$, choose $N \in \mathbb{N}$ such that

$$2^{-N-1} \leq \delta < 2^{-N}. \tag{4.3}$$

Then we have

$$\sup_{|t-s|<\delta} \frac{|f(t)-f(s)|}{\varphi(|t-s|)} \leq \sup_{|t-s|<2^{-N}} \frac{|f(t)-f(s)|}{\varphi(|t-s|)}. \tag{4.4}$$

For t,s satisfying $|t-s| < 2^{-N}$, there exists an $n \geq N$ such that $2^{-n-1} \leq |t-s| < 2^{-n}$. Thus it follows from (4.2) that for any t,s satisfying $|t-s| < 2^{-N}$,

$$\begin{aligned} |f(t)-f(s)| &\leq 3\Delta_n(f) \\ &\leq 3\varphi(|t-s|) \frac{\Delta_n(f)}{\varphi(2^{-n-1})} \\ &\leq 3\varphi(|t-s|) \sum_{n=N}^{\infty} \frac{\Delta_n(f)}{\varphi(2^{-n-1})}, \end{aligned}$$

implying

$$\sup_{|t-s|<2^{-N}} \frac{|f(t)-f(s)|}{\varphi(|t-s|)} \leq 3D_N(f), \tag{4.5}$$

where

$$D_N(f) = \sum_{n=N}^{\infty} \frac{\Delta_n(f)}{\varphi(2^{-n-1})}.$$

Note that $\delta\downarrow 0$ is equivalent to $N \longrightarrow \infty$ by (4.3). We thus conclude from (4.1), (4.4) and (4.5) that

$$\lim_{\delta\downarrow 0} \sup_{|t-s|<\delta} \frac{|f(t)-f(x)|}{\varphi(|t-s|)} \leq \lim_{N\to\infty} 3D_N(f) = 0. \qquad \square$$

Lemma 5. If (1.2) is satisfied, then

$$E[\sup_{0\le t\le 1} |X(t)-X(0)|^{\gamma}] \le \frac{K}{2} (\sum_{n=1}^{\infty} 2^{-n(\beta-1)/\gamma})^{\gamma}.$$

This is considered to be known, but we cannot find it in the literature. However, the proof can be found in the lecture note by one of the authors [K2] at National Taiwan University, which will be published.

<u>Lemma 6</u>. If (1.2) is satisfied, then for $0\le a<b\le 1$,

$$E[\sup_{a\le t\le b} |X(t)-X(s)|^{\gamma}] \le C(b-a)^{\beta}.$$

<u>Proof</u>. If we put $Y(t) := X(a+t(b-a))$, then for $0\le s,t\le 1$,

$$\begin{aligned} E[|Y(t)-Y(s)|^{\gamma}] &= E[|X(a+t(b-a)) - X(a+s(b-a))|^{\gamma}] \\ &\le K(b-a)^{\beta}|t-s|^{\beta}, \end{aligned}$$

under moment condition (1.2). Thus, by Lemma 5,

$$E[\sup_{a\le t\le b} |X(t)-X(a)|^{\gamma}] = E[\sup_{0\le t\le 1} |Y(t)-Y(0)|^{\gamma}] = C(b-a)^{\beta},$$

concluding the lemma. □

<u>Proof of Theorem 3</u>. Once again, we take a version X with continuous sample paths from the beginning.

As in Lemma 4, we put

$$\Delta_n(X) = \max_{0\le k\le 2^n} \sup_{0\le h\le 2^{-n}} |X(k2^{-n}+h) - X(k2^{-n})| .$$

We have

$$E[\sum_{n=1}^{\infty} \frac{\Delta_n(X)}{\varphi(2^{-n-1})}]$$

$$= \sum_{n=1}^{\infty} \frac{1}{\varphi(2^{-n-1})} E[\Delta_n(X)]$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{\varphi(2^{-n-1})} (E[\Delta_n(X)^{\gamma}])^{1/\gamma} \quad \text{(since } \gamma \geq 1\text{)}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{\varphi(2^{-n-1})} \Big(\sum_{n=1}^{2^n} E[\sup_{0 \leq h \leq 2^{-n}} |X(k2^{-n}+h)-X(k2^{-n})|^{\gamma}]\Big)^{1/\gamma}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{\varphi(2^{-n-1})} \Big(\sum_{n=1}^{2^n} C2^{-n\beta}\Big)^{1/\gamma} \quad \text{(by Lemma 6)}$$

$$= C \sum_{n=1}^{\infty} \frac{1}{\varphi(2^{-n-1})} (2^{-n(\beta-1)})^{1/\gamma}.$$

If we take here

$$\varphi(x) = |x|^{(\beta-1)/\gamma} \Big| \log |x| \Big|^{1+\epsilon}, \qquad \epsilon \geq 0,$$

then we have

$$E\Big[\sum_{n=1}^{\infty} \frac{\Delta_n(X)}{\varphi(2^{-n-1})} \Big] < \infty,$$

yielding

$$\sum_{n=1}^{\infty} \frac{\Delta_n(X)}{\varphi(2^{-n-1})} < \infty \quad \text{a.s.}$$

The conclusion follows from Lemma 4. □

References

[CM] S. Cambanis and M. Maejima, Two classes of self-similar stable processes with stationary increments, Stoch. Proc. Appl. 32 (1989), 305-329.

[K1] N. Kôno, On the modulus of continuity of sample functions of Gaussian processes, J. Math. Kyoto Univ. 10 (1970), 493-536.

[K2] N. Kôno, Recent Development of Random Fields and their Sample Paths, Lecture Note at National Taiwan University, 1989.

[KM] N. Kôno and M. Maejima, Self-similar stable processes with stationary increments, Proceedings of the Workshop on Stable Processes and Related Topics, 1990, to appear.

[MP] M.B. Marcus and G. Pisier, Characterization of almost surely continuous p-stable random Fourier series and strongly stationary processes, Acta. Math. 152 (1984), 245-301.

[T] K. Takashima, Sample path properties of ergodic self-similar processes, Osaka J. Math. 26 (1989), 159-189.

256. E. Mayer-Wolf, A central limit theorem in nonlinear filtering, Apr. 89.

257. C. Houdré, Factorization algorithms and non-stationary Wiener filtering, Apr. 89, Stochastics, to appear.

258. C. Houdré, Linear Fourier and stochastic analysis, Apr. 89.

259. G. Kallianpur, A line grid method in areal sampling and its connection with some early work of H. Robbins, Apr. 89. *Amer. J. Math. Manag. Sci.*, 1989, to appear.

260. G. Kallianpur, A.G. Miamee and H. Niemi, On the prediction theory of two-parameter stationary random fields, Apr. 89. *J. Multivariate Anal.*, 32, 1990, 120-149.

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